MATH 732: CUBIC HYPERSURFACES

DAVID STAPLETON

1. Some Classical Constructions

These notes are based on [Huy23, §1.5]. See the disclaimer section.

Example 1.1. Given a smooth cubic hypersurface:

$$X \subseteq \mathbf{P}^{n+1} = \mathbf{P}$$

we "saw in an exercise" that the maximum dimension of a linear subspace $\Lambda \subseteq X$ is $n/2$. It is not too hard to give examples with equality. For example, when n is even the *Fermat cubic*:

$$X = (x_0^3 + x_1^3 + \cdots x_n^3 + x_{n+1}^3 = 0) \subseteq \mathbf{P}$$

contains the linear subspace:

$$\Lambda = (x_0 + x_1 = x_2 + x_3 = x_n + x_{n+1} = 0).$$

This has codimension $n/2 + 1$ in $\mathbf{P}$ (so has dimension $n/2$). When n is odd, X contains the $(n-1)/2$ plane

$$(x_0 + x_1 = \cdots = x_{n-1} + x_n = x_{n+1} = 0) \subseteq X.$$

Example 1.2. If $\mathbf{P} = \mathbf{P}(V)$ and $\Lambda = \mathbf{P}(W)$, then the rational map *linear projection from* Λ

$$q_\Lambda : \mathbf{P} \dashrightarrow \mathbf{P}(V/W) = \mathbf{P}'$$

is induced by the linear quotient $q_W : V \to V/W$ and sends a one-dimensional subspace $\lambda \subseteq V \mapsto q(\lambda)$ as long as $\lambda \notin W$. I.e. the base locus of this map is $\Lambda \subseteq \mathbf{P}$. For a one-dimensional subspace $\lambda \in \mathbf{P} \smallsetminus \Lambda$, the closure of the fiber of q_Λ at λ is the linear subspace $\mathbf{P}(W + \lambda) \subseteq \mathbf{P}$. Likewise, any linear subspace of $\mathbf{P}$ having dimension $\dim(\Lambda) + 1$ that contains Λ is the closure of a fiber of p_Λ. The closure of the graph of p_Λ:

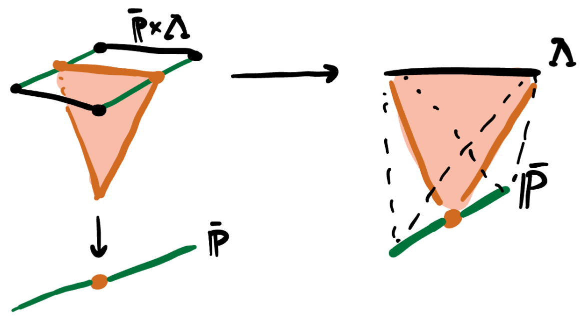
$$\Gamma \subseteq \mathbf{P} \times P(V/W).$$

is the blow-up of $\mathbf{P}$ at Λ. If $\mu : \Gamma \to \mathbf{P}$ is the blow-up map, then the projection:

$$\Gamma \to \mathbf{P}(V/W)$$

is associated to the complete linear system $|\mu^*\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{O}_{\mathbf{P}}(-E)|$. The map $\phi\colon \Gamma \to \mathbf{P}(V/W)$ corresponds to the projective bundle:

$$\Gamma \simeq \mathbf{P}(\mathcal{O}_{\mathbf{P}(V/W)}(1) \oplus \mathcal{O}_{\mathbf{P}(V/W)}^{\oplus \dim W}).$$



For a smooth cubic hypersurface $X = (F = 0)$ containing Λ, the cubic equation pulls back to a section

$$\mu^* F \in \mathrm{H}(\Gamma, \mu^*\mathcal{O}(3)).$$

As X has multiplicity 1 along Λ, so $\mu^* F$ gives rise to a section of

$$\mu^* F \in \mathrm{H}(\Gamma, \mu^*\mathcal{O}(3) \otimes \mathcal{O}(-E)).$$

This corresponds to the blowing up of X at Λ. From the perspective of the projective bundle ϕ, this gives a section of

$$\mathcal{O}_\phi(2) \otimes \phi^*(\mathcal{O}_{\mathbf{P}(V/W)}(1)),$$

which is to say that $\mu^* F$ is a family of quadrics in the fiber of Γ. Explicitly, given a $\dim(\Lambda) + 1$ plane Pi containing Λ, we know that (if Π meets X properly) Π meets X at a degree 3 hypersurface in Π:

$$\Pi \cap X = Q_\Pi \cup \Lambda \subseteq \Pi.$$

The quadric Q_Π is called the *residual quadric*.

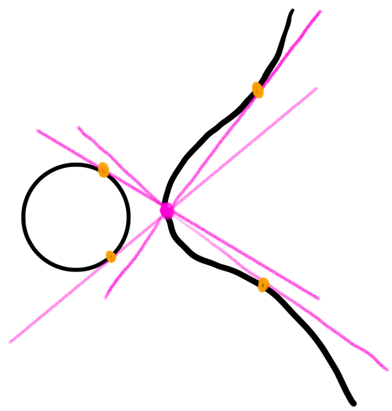
If we let $\mathcal{E} = \mathcal{O}_{\mathbf{P}'}(1) \oplus \mathcal{O}_{\mathbf{P}'}^{\oplus \dim W}$, then $\phi_*\mu^* F$ gives a section of $\mathrm{Sym}^2(\mathcal{E})(1)$, or equivalently a symmetric homomorphism:

$$\phi_*\mu^* F\colon \mathcal{E}^* \to \mathcal{E}(1).$$

We have shown that X is birationally a quadric bundle over $\mathbf{P}(V/W)$. The singular fibers correspond to when the map $\phi_*\mu^* F$ becomes singular, which is when $\det(\phi_*\mu^* F) = 0$. This is a section of the line bundle

$$\det(\mathcal{E}) \otimes \det(\mathcal{E}(1)) \simeq \det(E)^2 \otimes \mathcal{O}_{\mathbf{P}'}(\dim W + 1) \simeq \mathcal{O}_{\mathbf{P}'}(\dim W + 3).$$

Example 1.3. If we project from a point on a smooth cubic hypersurface (so $\dim W = 1$) this gives a double cover of $\mathbf{P}(V/W)$ that is branched along a degree 4 hypersurface. (Likewise, if we project from a line this gives a conic bundle over $\mathbf{P}(V/W)$ that is branched along a quintic.)



Exercise 1. *Let*

$$X = (x_0^3 + \cdots + x_{n+1}^3 = 0) \subseteq \mathbf{P}$$

be an even dimensional Fermat cubic hypersurface and let

$$\Lambda = (x_0 + x_1 = \cdots = x_n + x_{n+1} = 0) \subseteq \mathbf{P}.$$

Show that the corresponding quadric fibration is singular along the union of $n/2+1$ *hyperplanes and the cubic hypersurface:*

$$X \cap (x_0 - x_1 = \cdots x_n - x_{n+1} = 0)$$

thought of as a subset of $\mathbf{P}^{n/2}$.

Example 1.4. If $\mathbf{P}^2 \simeq \Lambda \subseteq X \subseteq \mathbf{P}^5$ is a cubic fourfold that contains a plane then we can use quadric fibration to prove that X is *unirational* (i.e. X admits a dominant map from projective space). To do this, we choose an auxiliary $\mathbf{P}^3 \subseteq \mathbf{P}^5$. This meets X at a smooth, rational cubic surface, which double covers $\mathbf{P}'$. The base change of the quadric bundle to the cubic surface is rational because it's a quadric bundle over a rational surface with a point. This gives a degree 2 unirational parametrization.

Example 1.5 (Rational Hypersurfaces)**.** If X is a smooth, even dimensional cubic hypersurface of dimension n that contains *two* complementary $n/2$-dimensional linear subspaces $\Lambda_1, \Lambda_2 \subseteq X$ that span $\mathbf{P}$, then X is even rational! The *third point map*:

$$\Lambda_1 \times \Lambda_2 \dashrightarrow X$$

where a pair of points (λ_1, λ_2) maps to the third point on the line $\overline{\lambda_1\lambda_2} \cap X$.

Example 1.6 (A general unirationality construction)**.** We know cubic surfaces are rational. This lets us inductively prove cubic hypersurfaces are rational. Consider a smooth cubic hypersurface X with two hyperplane sections Y_1 and Y_2. Then the *third point map*

$$Y_1 \times Y_2 \dashrightarrow X$$

gives a dominant map to X. As Y_1 and Y_2 are unirational, the product is also unirational, which does the job.

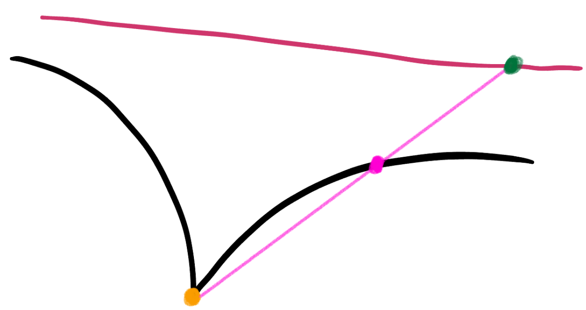
Example 1.7 (Rationality of nodal cubics)**.** Suppose that $X \subseteq \mathbf{P}$ is a reduced, irreducible cubic with a double point. Projection from this point gives a map

$$X \dashrightarrow \mathbf{P}^n$$

of degree 1, which shows the cubic is rational. We can likewise parametrize the points on a cubic via a third point construction. Let $\mathbf{P}^n \simeq \Pi \subseteq \mathbf{P}$ be a linear subspace that does not contain the double point $p \in X$. Then, there is a rational map:

$$\Pi \dashrightarrow X$$

that sends a point $y \in \Pi$ to the final point on the line $\overline{py} \cap X$.



Moreover, if $p \in X = (F = 0)$ is an *ordinary double point* and is the only singular point, we can understand what gets contracted by the birational map $X \dashrightarrow \mathbf{P}^n$. For simplicity assume that $p = [0 : \cdots : 0 : 1] \in \mathbf{P}$. Then, we can expand the equation F as

$$F = Q(x_0, \ldots, x_n)x_{n+1} + G(x_0, \ldots, x_n)$$

where $Q(x_0, \cdots, x_n)$ is a non-degenerate quadric and G is a homogeneous equation in one fewer variables. (Note that $x_{n+1} \neq 0$ at p.) The complete intersection $D = (Q = G = 0)$ is a divisor in X, which is a cone over a subvariety in $\mathbf{P}^n = (x_{n+1} = 0)$. The assumption that X is smooth away from p implies this complete intersection is smooth in $\mathbf{P}^n$. If

$$X' \subseteq \mathbf{P} \times \mathbf{P}^n$$

is the graph of this birational map then the projection $X' \to \mathbf{P}^n$ corresponds to the blow-up of the $(2,3)$ complete intersection variety, and the projection $X' \to \mathbf{P}$ corresponds to the contraction of the quadric $Q = 0$. To be more explicit: let X be a cubic with a unique singularity that is an ODP:

(1) If X is a surface, then it corresponds to the blow-up of 6 points in $\mathbf{P}^2$ that are the intersection of a conic and a cubic, followed by the contraction of the conic.
(2) If X is a cubic threefold, then it corresponds to the blow-up of a canonical genus 4 curve $C \subseteq \mathbf{P}^3$ followed by the contraction of the unique conic that contains it.
(3) If X is a cubic fourfold, then it corresponds to the blow-up of a (2,3) complete intersection K3 surface $S \subseteq \mathbf{P}^4$, followed by the contraction of the unique quadric containing it.

Example 1.8. It is also interesting to ask: *What is the maximal number δ of ordinary double points a cubic hypersurface can have?* Roughly speaking, this should correspond to a normal crossing singularity of $D(3,n)$ with δ crossings. For cubics this is known to be:

$$\binom{n+2}{\lfloor (n+1)/2 \rfloor}.$$

For example, when $n = 2$ this gives 4 and when $n = 3$ we get 10. These are uniquely given by the famous *Cayley surface*:

$$\left(x_0 \cdots x_3 \left(\frac{1}{x_0} + \cdots + \frac{1}{x_3} \right) = 0 \right) \subseteq \mathbf{P}^3$$

and the *Segre cubic threefold*:

$$\left(\sum_{i=0}^{5} x_i^3 = \sum_{i=0}^{5} x_i = 0 \right) \subseteq \mathbf{P}^4 \subseteq \mathbf{P}^5.$$

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.